World Nematic Crystal Model of Gravity Explaining the Absence of Torsion

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We attribute the gravitational interaction between sources of curvature to the world being a crystal which has undergone a quantum phase transition to a nematic phase by a condensation of dislocations. The model explains why spacetime has no observable torsion and predicts the existence of curvature sources in the form of world sheets, albeit with different high-energy properties than those of string models.

I. INTRODUCTION

Present-day string models of elementary particles are based on the assumption that relativistic physics will prevail at all energy scales and, moreover, show recurrent particle spectra at arbitrary multiples of the Planck mass. Disappointed by the failure of these models [1] to explain correctly even the the low-lying excitations, and the apparent impossibility of ever observing the characteristic recurrences, an increasing number of theoreticians is beginning to suspect that God may have chosen a completely different extension of present-day Lorentz-invariant physics to extremely high energies [2, 3, 4]. This philosophy has been advocated by one of the authors (HK) for almost two decades. In 1987, he proposed a simple three-dimensional euclidean world crystal model of gravitation in which dislocations and disclinations represent curvature and torsion in the geometry of spacetime [5]. A full theory of gravity with torsion based on this picture is published in the textbook [6] (see also [7, 8]).

The simple 1987 model had the somewhat unaesthetic feature that the crystal possessed only second-gradient elasticity to deliver the correct forces between the sources of curvature, which for an ordinary first-gradient elasticity grow linearly with the distance R and are thus confining. In this note we would like to point out that the correct $1/R$-behavior can also be obtained in an ordinary first-gradient world crystal with first-gradient elasticity by assuming that the dislocations have proliferated. This explains also why the theory of general relativity requires only curvature for a correct description of gravitational forces, but no torsion. Such a state of the world crystal bears a close relationship with the nematic quantum liquid crystals of condensed matter physics, first suggested by Kivelson et al. [9], and believed to be of relevance both for the quantum Hall effect [10] and in high-T_c superconductors [11].

Our model will be formulated as before in three euclidean dimensions, for simplicity. The generalization to four dimensions is straightforward. The elastic energy is expressed in terms of a material *displacement field* $u_i(\mathbf{x})$ as

$$E=\int d^3x\,\left[\mu u_{ij}^2(\mathbf{x})+\frac{\lambda}{2}u_{ii}^2(\mathbf{x})\right], \tag{1.1}$$

where

$$u_{ij}(\mathbf{x})\equiv\frac{1}{2}[\partial_i u_j(\mathbf{x})+\partial_j u_j(\mathbf{x})] \tag{1.2}$$

is the *strain tensor* and μ,ν are the elastic the shear moduli. The elastic energy goes to zero for infinite wave length since in this limit $u_i(\mathbf{x})$ reduces to a pure translation under which the energy of the system is invariant. The crystallization process causes a spontaneous breakdown of the translational symmetry of the system. The elastic distortions describe the Nambu-Goldstone modes resulting from this symmetry breakdown. Note that so far the crystal has an extra longitudinal sound wave with a different velocity than the shear waves.

A crystalline material always contains defects. In their presence, the elastic energy is

$$E=\int d^3x\,\left[\mu(u_{ij}-u_{ij}^{\mathrm{p}})^2+\frac{\lambda}{2}(u_{ii}-u_{ii}^{\mathrm{p}})^2\right], \tag{1.3}$$

where u_{ij}^{p} is the so-called *plastic strain tensor* describing the defects. It is composed of an ensemble of lines with a dislocation density

$$\alpha_{il}=\epsilon_{ijk}\partial_j\partial_k u_l(\mathbf{x})=\delta_i(\mathbf{x};L)(b_l+\epsilon_{lqr}\Omega_q x_r). \tag{1.4}$$

and a *disclination density*

$$\theta_{il}=\epsilon_{ijk}\partial_j\phi_{kl}^{\mathrm{p}}=\delta_i(\mathbf{x};L)\Omega_l, \tag{1.5}$$

where b_l and Ω_l are the so-called Burgers and Franck vectors of the defects. The densities satisfy the conservation laws

$$\partial_i\alpha_{ik}=-\epsilon_{kmn}\theta_{mn},\quad \partial_i\theta_{il}=0. \tag{1.6}$$

Dislocation lines are either closed or they end in disclination lines, and disclination lines are closed. These are Bianchi identities of the defect system.

An important geometric quantity characterizing dislocation and disclination lines is the *incompatibility* or *defect density*

$$\eta_{ij}(\mathbf{x}) = \epsilon_{ikl}\epsilon_{jmn}\partial_k\partial_m u^P_{ln}(\mathbf{x}). \tag{1.7}$$

It can be decomposed into disclination and dislocation density as follows [6]:

$$\eta_{ij}(\mathbf{x})=\theta_{ij}(\mathbf{x})+\frac{1}{2}\partial_m\left[\epsilon_{min}\alpha_{jn}(\mathbf{x})+(i\leftrightarrow j)-\epsilon_{ijn}\alpha_{mn}(\mathbf{x})\right]. \tag{1.8}$$

This tensor is symmetric and conserved

$$\partial_i\eta_{ij}(\mathbf{x}) = 0, \tag{1.9}$$

again a Bianchi identity of the defect system.

It is useful to separate from the dislocation density (1.4) the contribution from the disclinations which causes the nonzero right-hand side of (1.6). Thus we define a *pure dislocation density*

$$\alpha^b_{ij}(\mathbf{x}) \equiv \alpha_{ij}(\mathbf{x}) - \alpha^\Omega_{ij}(\mathbf{x}) \tag{1.10}$$

which satisfies $\partial_i\alpha^b_{ij} = 0$. Accordingly, we split

$$\eta_{ij}(\mathbf{x}) = \eta^b_{ij}(\mathbf{x}) + \eta^\Omega_{ij}(\mathbf{x}), \tag{1.11}$$

where

$$\eta^b_{ij}(\mathbf{x})=\frac{1}{2}\left[\epsilon_{min}\alpha^b_{jn}(\mathbf{x}) + (i\leftrightarrow j) - \epsilon_{ijn}\alpha^b_{mn}(\mathbf{x})\right], \tag{1.12}$$

and the pure disclination part of the defect tensor looks like (1.8), but with superscripts Ω on η_{ij} and α_{ij}.

The tensors α_{ij}, θ_{ij}, and η_{ij} are linearized versions of important geometric tensors in the *Riemann-Cartan space* of defects, a noneuclidean space with curvature and torsion. Such a space can be generated from a flat space by a plastic distortion, which is mathematically represented by a *nonholonomic* mapping [7, 8] $x_i \to x_i+u_i(\mathbf{x})$. Such a mapping is nonintegrable. The displacement fields and their first derivatives fail to satisfy the Schwarz integrability criterion:

$$(\partial_i\partial_j-\partial_j\partial_i)\,u(\mathbf{x}) \neq 0, \quad (\partial_i\partial_j-\partial_j\partial_i)\,\partial_k u_l(\mathbf{x}) \neq 0. \tag{1.13}$$

The metric and the affine connection of the geometry in the plastically distorted space are $g_{ij} = \delta_{ij} + \partial_i u_j + \partial_j u_i$ and $\Gamma_{ijl} = \partial_i\partial_j u_l$, respectively. The noncommutativity of the derivatives in front of $u_l(\mathbf{x})$ implies a nonzero torsion, the torsion tensor being $S_{ijk} \equiv (\Gamma_{ijk}-\Gamma_{jik})/2$. The dislocation density α_{ij} is equal to $\alpha_{ij} = \epsilon_{ikl}S_{klj}$,

The noncommutativity of the derivatives in front of $\partial_k u_l(\mathbf{x})$ implies a nonzero curvature. The disclination density θ_{ij} is the Einstein tensor $\theta_{ij} = R_{ji} - \frac{1}{2}g_{ji}R$ of this Einstein-Cartan defect geometry. The tensor η_{ij}, finally, is the Belinfante symmetric energy momentum tensor, which is defined in terms of the canonical energy-momentum tensor and the spin density by a relation just like (1.8). For more details on the geometric aspects see Part IV in Vol. II of [6], where the full one-to-one correspondence between defect systems and Riemann-Cartan geometry is developed as well as a gravitational theory based on this analogy.

Let us now show how linearized gravity emerges from the energy (1.3). For this we eliminate the jumping surfaces in the defect gauge fields from the partition function by introducing conjugate variables and associated stress gauge fields. This is done by rewriting the elastic action of defect lines as

$$E=\int d^3x\left[\frac{1}{4\mu}\left(\sigma^2_{ij}-\frac{\nu}{1+\nu}\sigma^2_{ii}\right)+i\sigma_{ij}(u_{ij}-u^P_{ij})\right], \tag{1.14}$$

where $\nu \equiv \lambda/2(\lambda+\mu)$ is Poisson's ratio, and forming the partition function, integrating the Boltzmann factor e^{-E/k_BT} over σ_{ij}, u_i, and summing over all jumping surfaces S in the plastic fields. The integrals over u_i yield the conservation law $\partial_i\sigma_{ij} = 0$. This can be enforced as a Bianchi identity by introducing a stress gauge field h_{ij} and writing $\sigma_{ij} = G_{ij} \equiv \epsilon_{ikl}\epsilon_{jmn}\partial_k\partial_m h_{ln}$. The double curl on the right-hand side is recognized as the Einstein tensor in the geometric description of stresses, expressed in terms of a small deviation n $h_{ij} \equiv g_{ij}-\delta_{ij}$ of the metric from the flat-space form. Inserting G_{ij} into (1.14) and using (1.7), we can replace the energy in the partition function by $E = E^{\rm stress} + E^{\rm def}$ where

$$E^{\rm stress}+E^{\rm def}\equiv\int d^3x\left[\frac{1}{4\mu}\left(G^2_{ij}-\frac{\nu}{1+\nu}G^2_{ii}\right)+ih_{ij}\eta_{ij}\right], \tag{1.15}$$

where the defect tensor (1.8) has the decomposition

$$\eta_{ij} = \eta^\Omega_{ij} + \partial_m\epsilon_{min}\alpha^b_{jn}. \tag{1.16}$$

The defects have also core energies which has been ignored so far. Adding these for the dislocations and ignoring, for a moment, the disclination part of the defect density in (1.16), we obtain

$$E^{\rm disl}=i\int d^3x\left(\epsilon_{imn}\partial_m h_{ij}\alpha^b_{jn} + \frac{\epsilon_c}{2}\alpha^{b\,2}_{jn}\right). \tag{1.17}$$

We now assume that the world crystal has undergone a transition to a condensed phase in which dislocations are condensed. To reach such a state, whose existence was conjectured for two-dimensional crystals in Ref. [12], the model requires a modification by an additional rotational energy, as shown in [13] and verified by Monte Carlo simulations in [14]. The three-dimensional extension of the model is described in [6].

The condensed phase is described by a partition function in which the discrete sum over the pure dislocation densities in α^b_{jn} is approximated by an ordinary functional integral. This has been shown in Ref. [7]. The general integration rule is

$$\int d^3l\,\delta(\boldsymbol{\partial}\cdot\mathbf{l})\,e^{-\beta\mathbf{l}^2/2+i\mathbf{l}\mathbf{a}} = e^{-\mathbf{a}^2_T/2\beta}, \tag{1.18}$$

where $\mathbf{a}_T$ has the components $a_{Ti} \equiv -i\epsilon_{ijk}\partial_j a_k/\sqrt{-\boldsymbol{\partial}^2}$. The Boltzmann factor resulting in this way from $E^{\rm stress}$ plus (1.17) has now the energy

$$E' = \int d^3x \left[\frac{1}{4\mu}\left(G_{ij}^2 - \frac{\nu}{1+\nu}G_{ii}^2\right) + \frac{1}{2\epsilon_c}G_{ij}\frac{1}{-\boldsymbol{\partial}^2}G_{ij}\right]. \tag{1.19}$$

The second term implies a Meissner-like screening of the initially confining gravitational forces between the disclination part of the defect tensor to Newton-like forces. For distances longer than the Planck scale, we may ignore the stress term and find the effective gravitational action for the disclination part of the defect tensor:

$$E \approx \int d^3x \left(\frac{1}{2\epsilon_c}G_{ij}\frac{1}{-\boldsymbol{\partial}^2}G_{ij} + ih_{ij}\eta_{ij}^{\Omega}\right). \tag{1.20}$$

A path integral over h_{ij} and a sum over all line ensembles applied to the Boltzmann factor $e^{-E/\hbar}$ is a simple Euclidean model of pure quantum gravity. The line fluctuations of η_{ij}^{Ω} describe a fluctuating Riemann geometry perforated by a grand-canonical ensemble arbitrarily shaped lines of curvature. As long as the loops are small they merely renormalize the first term in the energy (1.20). Such effects were calculates in closely related theories in great detail in Ref. [21]. They also give rise to post-Newtonian terms in the above linearized description of the Riemann space.

We may now add matter to this gravitational environment. It is coupled by the usual Einstein interaction

$$E^{\rm int} \approx \int d^3x\, h_{ij}T^{ij}, \tag{1.21}$$

where T^{ij} is the symmetric Belinfante energy momentum tensor of matter. Inserting for G_{ij} the double-curl of h_{ij} we see that the energy (1.20) produces the correct Newton law if the core energy is $\epsilon_c = 8\pi G$, where G is Newton's constant.

Note that the condensation process of dislocations has led to a pure Riemann space without torsion. Just as a molten crystal shows residues of the original crystal structure only at molecular distances, remnants of the initial torsion could be observed only near the Planck scale. This explains why present-day general relativity requires only a Riemann space, not a Riemann-Cartan space.

In the non-relativistic context, a dislocation condensate is characteristic for a nematic liquid crystal, whose order is translationally invariant, but breaks rotational symmetry (see [6, 12] in the two dimensions and [15] in the 2+1-dimensional quantum theory). The Burgers vector of a dislocation is a vectorial topological charge, and nematic order may be viewed as an ordering of the Burgers vectors in the dislocation condensate. Such a manifest nematic order would break the low energy Lorentz-invariance of space-time. We may, however, imagine that the stiffness of the directional field of Burgers vectors is so low that, by the criterion of Ref. [16], they have undergone a Heisenberg-type of phase transition into a directionally disordered phase in an environment with only a few disclinations. In three dimensions, dislocations (and disclinations) are line-like. This has the pleasant consequence, that they can be described by the disorder field theories developed in [17] in which the proliferation of disclinations follows the typical Ginzburg-Landau pattern of the field expectation acquiring a nonzero expectation value. A cubic interaction becomes isotropic in the continuum limit [18] (this is the famous fluctuation-induced symmetry restoration of the Heisenberg fixed point in a ϕ^4-theory with O(3)-symmetric plus cubic interactions [19]. The isotropic phase is similar to what has been called a *topological* form of nematic order by Lammert et al. [20] in a generalized Z_2 gauge theory of nematic order. Also there rotational (Lorentz-) invariance is restored even though there is no condensate of disclinations, and the Burgers vectors are disordered by fluctuations (see also ref. [15]). The dual of this phase is Coulomb-like.

The above description of defects was formulated in what has been named tangential approximation to the Euclidean group [6], in which the discrete rotations are treated as if they took place in the tangential place with arbitrary real Franck vectors Ω^i [see Eq. (1.4)]. In a more accurate formulation, the nonabelian nature of the rotations and the quantization of Ω_i must be taken into account. Their discreteness is certainly remembered in the nematic phase, even if the directions of the Burgers vectors become diordered (see also the discussion in Ref. [22]). This implies that there are elements of quantized curvature fluctuating in spacetime. Fortunately, this does not introcduce any unphasical results at presently accessible length scales since these fluctuations mainly renormalize the basic curvature energy in (1.20), as discussed before.

The generalization to four Euclidean spacetime dimensions changes mainly the geometry of the defects. In four dimensions, they become world sheets, and a second-quantized disorder field description of surfaces has not yet been found. But the approximation of representing a sum over dislocation surfaces in the proliferated phase as an integral as in Eq. (1.18) will remain valid, so that the above line of arguments will survive, this being a natural generalization of the Meissner-Higgs mechanism. The disclination sources of curvature will be world sheets, as an attractive feature for string theorists. However, the high-energy properties will be completely different. On the one hand, these surfaces behave nonrelativistically as the energies approach the Planck scale, on the other hand they will not have the characteristic multi-Planck recurrences of the common strings. Although the latter property may never be verified in the laboratory, the deviations from relativity at high energies or short distances may come into experimentalists reach in the not too distant future.

Note that our model has automatically a vanishing cos-

mological constant. Since the atoms in the crystal are in equilibrium, the pressure is zero. This explanation is similar to that given by Volovik [2] with his helium droplet analogies.

[1] For a summary of failures see T. Banks, *A Critique of Pure String Theory: Heterodox Opinions of Diverse Dimensions*, (hep-th/0306074)

[2] G.E. Volovik, Phys. Rep. **351**, 195-348 (2001) (gr-qc/0005091); *Phenomenology of effective gravity*, gr-qc/0304061

[3] G. Chapline, E. Hohlfeld, R. B. Laughlin, D. I. Santiago, *Quantum Phase Transitions and the Breakdown of Classical General Relativity*, gr-qc/0012094.

[4] S.Z. Zhang, *To see a world in a grain of sand*, hep-th/0210162.

[5] H. Kleinert, *Gravity as Theory of Defects in a Crystal with Only Second-Gradient Elasticity*, Ann. d. Physik, **44**, 117 (1987) (http://www.physik.fu-berlin.de/~kleinert/172)

[6] H. Kleinert, *Gauge fields in Condensed Matter*, Vol. I: *Superflow and Vortex Lines, Disorder Fields, Phase Transitions*, Vol. II: *Stresses and Defects, Differential Geometry, Crystal Defects* , World Scientific, Singapore, 1989. (http://www.physik.fu-berlin.de/~kleinert/b1)

[7] H. Kleinert, *Theory of Fluctuating Nonholonomic Fields and Applications: Statistical Mechanics of Vortices and Defects and New Physical Laws in Spaces with Curvature and Torsion*, in: Proceedings of NATO Advanced Study Institute on Formation and Interaction of Topological Defects at the University of Cambridge, Plenum Press, New York, 1995, S. 201–232 (cond-mat/9503030)

[8] H. Kleinert, *Nonholonomic Mapping Principle for Classical and Quantum Mechanics in Spaces with Curvature and Torsion,*, Gen. Rel. Grav. **32** , 769 (2000) (http://www.physik.fu-berlin.de/~kleinert/258/258j.pdf)

[9] S.A. Kivelson, E. Fradkin, and V.J. Emery, Nature **393**, 550 (1998).

[10] E. Fradkin, S.A. Kivelson, E. Manousakis, and K. Nho, Phys. Rev. Lett. **84**, 1982 (2000).

[11] J. Zaanen, Science **286**, 251 (1999) and references therein.

[12] B.I. Halperin and D.R. Nelson, Phys. Rev. Lett. **41**, 121 (1978).

[13] H. Kleinert, *Lattice Defect Model with Two Successive Melting Transitions*, Phys. Lett. A **130**, 443 (1988) (http://www.physik.fu-berlin.de/~kleinert/174).

[14] W. Janke and H. Kleinert, *From First-Order to Two Continuous Melting Transitions - Monte Carlo Study of New 2D Lattice Defect Model*, Phys. Rev. Lett. **61**(20), 2344(1988) (http://www.physik.fu-berlin.de/~kleinert/179).

[15] J. Zaanen, S.I. Mukhin, and Z. Nussinov, in preparation.

[16] H. Kleinert, *Criterion for Dominance of Directional versus Size Fluctuations of Order Field in Restoring Spontaneously Broken Continuous Symmetries* Phys. Rev. Lett. **84** , 286 (2000) (cond-mat/9908239).

[17] For the derivation of this connection wee pp. 133–164 in Vol. I of the textbook [6].

[18] See the discussion on pp. 938–952 in Vol. I of the textbook [6].

[19] See Chapter 18 in H. Kleinert and V. Schulte-Frohlinde, Critical Properties of Φ^4-Theories, World Scientific, Singapore 2001, pp. 1–487 (http://www.physik.fu-berlin.de/~kleinert/b8)

[20] P.E. Lammert, D.S. Rokhsar, and J. Toner, Phys. Rev. E **52**, 1778 (1995).

[21] H. Kleinert and W. Miller, Phys. Rev. Lett. **56**, 11 (1986); Phys. Rev. D **38**, 1239(1988).

[22] N.M. Muller, Thesis, University of Amsterdam (1998); N.M. Muller and F.A. Bais (unpublished).